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SUGGESTIONS FOR A MATHEMATICAL BIOPHYSICS OF SOME PSYCHOSES

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THE UNIVERSITY OF CHICAGO

We may consider that most of the human behavior is a set of learned responses to certain patterns which recur frequently in the course of human life. Some "abnormal" events or experiences may result in the learning of abnormal responses, and thus in abnormal behavior. The "abnormal" responses may begin to be learned after some of the normal response patterns have been fairly well established. The development of both normal and abnormal behavior may thus be represented by learning curves of the type studied by H. D. Landahl. Applying some of the results of the theory of learning curves and considering that the normal and abnormal reactions may reciprocally inhibit each other, a quantitative theory of some psychoses may be developed. In particular, the effects of shock may be deduced from the assumption that they cause the more recently learned abnormal reactions to be "unlearned" more readily, than the earlier learned "normal" reactions. The effectiveness of shock treatments as a function of the duration of psychosis is discussed from this point of view.

In a recent paper L. Danziger and H. D. Landahl (1945) have outlined a mathematical theory of the effects of shock treatments in schizophrenia to account for certain qualitative regularities, observed by L. Danziger, in the relation between the number of shocks, duration of illness before treatment, and percentage of successful treatments. The theory is purely formal and while suggestive for further studies does not consider any specific mechanism, either neurological or psychological. An interesting paper on the mathematical theory of psychoses by J. Lettvin and W. Pitts (1943) is also of a purely formal nature. It is the purpose of this note to outline a neurobiophysical mechanism for some psychoses, and perhaps some psychoneuroses, which would in a general way account for the empirical relations discussed by L. Danziger and H. D. Landahl as well as suggest some new quantitative experiments.

We may consider that most, if not all of our behavior, is a set of learned, or conditioned, responses to certain patterns which recur frequently in the course of our lives. The average life-patterns of average individuals being fairly much alike, the responses of an average individual to a given situation are also nearly the same for all average or "normal" individuals. In the course of a socially normal

life, we do learn to respond to various situations in a manner which is to our best advantage in normal society, and such responses result in normal social adjustments of the individual.

On the other hand, abnormal situations may result in learning a set of wrong reactions. Thus, to take a trivial example, while the normal reaction of a normal individual to others may be a general open-minded friendliness, an individual who was brought up with people who maltreated him may react to all individuals with suspicion, fear, or hatred.

The "correctness" of a response pattern is not to be judged only from the point of view of social adjustment. We also learn to respond to a number of external situations. We do learn to think correctly, to generalize from certain observations, etc.

Assuming a biophysical mechanism of learning suggested by N. Rashevsky (1938) and elaborated in detail with several applications by H. D. Landahl (1941a, b; 1943), we find that the learning curve is monotonically increasing with the number of trials, or which is the same, with time. It tends asymptotically to a constant value. The ordinates of the curve represent the percentage of correct responses after a given time of training. When the percentage is practically 100%, learning is complete, and further training does not improve the performance.

The behavior patterns of an individual show in general the same characteristics. In childhood we react frequently to a situation in a manner which is socially inadequate, hence "wrong". As we grow, the frequency of "correct" responses to a given situation increases.

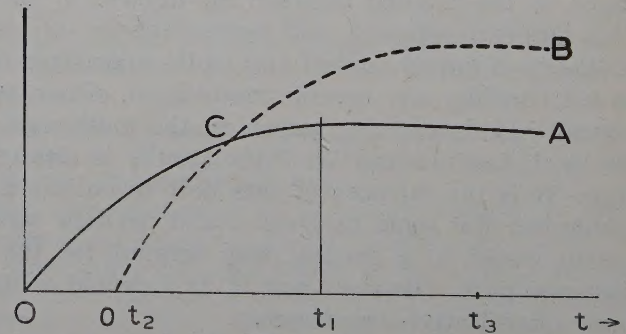

FIGURE 1

The learning curve may be viewed also from a different point. It may represent the intensity of a graded response to a stimulus, as a function of learning time. This is, for instance, the case with the development of the conditioned salivary reflex.

Let the curve OA (Figure 1) represent the learning curve of the general behavior of an individual, as a function of the training time t. The time t in this case roughly coincides with the individual's age. Let us first consider the ordinates as representing the intensities of a graded reaction.

Suppose that at a time t_1, when the individual has already well learned the correct responses, he is put into an abnormal situation which, if lasting long enough, would result in his learning a wrong response to certain standard patterns. If, as is usually assumed (Rashevsky, 1938), the efferent pathways which correspond to different responses cross-inhibit each other, then the already strongly-developed correct response will inhibit the newly-learned weak response, and nothing will happen unless the abnormal situation contains an unusually strong set of stimuli which tend to produce the wrong response.

But if a similar situation occurs at an earlier age, t_2, (Figure 1), when the correct responses have not been thoroughly learned, then the intensity of the correct response may not be sufficient to inhibit completely the wrong response. If the abnormal situation lasts for a sufficiently long time, a wrong response pattern will develop along a curve $O'B$. The intensity of the wrong response may or may not eventually exceed that of the correct one.

Since the external environment of an individual is largely moulded by his own reactions, a situation may arise in which the very fact that the individual becomes conditioned to wrong responses will prolong the abnormal situations, which favors that conditioning. In this way, the process becomes self-perpetuating.

Approximately we may say that as long as the intensity R_c of the correct response is stronger than the intensity R_w of the wrong one, the individual behaves "normally". Actually, the situation is more complicated. Due to fluctuations of thresholds and of the excitatory and inhibitory factors, even if $R_c > R_w$, a certain percentage of responses will be wrong (Landahl, 1938; Householder and Landahl, 1945). The frequency P_w of incidence of the wrong response is a function $f(R_c, R_w)$ of R_c and R_w. Expressions for that functional relationship have been derived by H. D. Landahl (1938). Therefore, even if the curve $O'B$ remains always below OA, the individual will show a certain percentage of "wrong" responses. In fact, as is actually the case, an individual who on the average is perfectly "normal" may once in a while do some "queer" things. It is largely a matter of social and clinical convention as to how large P_w should become in order to consider the individual as "abnormal".

If one would introduce arbitrarily a certain criterion of abnor-

mality, let us say $P_w \gtreqqless 0.25$, and make in the clinic elaborate *qualitative statistical* observations on the behavior of each individual, we not only would have a *quantitative measure* of a person's "abnormality" P_w (or "normality"), but from the relation

$$P_w = f(R_c, R_w) \tag{1}$$

we may draw some quantitative conclusions as to the difference in intensities of the "normal" and "abnormal" states, which in this case may be considered as being measured by the ordinates of the curves OA and $O'B$ of Figure 1.

When $R_c - R_w$ is very small, then in general (Landahl, 1938; Householder and Landahl, 1945) in a certain number of situations no response at all will occur. If such a situation is of lasting duration, we shall have a behavior reminiscent of stupors and catatonia. In general, there is a finite probability P_e of incidence of "no response" and P_c of correct response, so that

$$P_w + P_c + P_e = 1 . \tag{2}$$

Moreover, there are relations (Landahl, 1938; Householder and Landahl, 1945) between the three quantities P_w, P_c, and P_e. If these three quantities are determined for each individual experimentally, then those relations should hold for the population as a whole, assuming *average* constant values for all other neurobiophysical parameters. Here is something worth trying as a quantitative approach to some psychiatric problems.

If we interpret some psychoses, and perhaps psychoneuroses, as the result of "wrong" conditioning, we see why this type of mental disease is much more likely to start at a relatively young age. If, from considerations of mathematical sociology, we can compute the probability of occurence of abnormal situations, and if the equations of the curve OA and $O'B$ are known, we can determine theoretically the probability of the beginning of a psychosis at a given age, and compare it with available statistical data.

The equation of the learning curve, when the latter represents the intensity of the response, is essentially determined by the integral curve of the distribution function

$$N(h)\,dh \tag{3}$$

of the hysteresis thresholds of the self-circuited pathways responsible for conditioning (Rashevsky, 1938). For each value of h a certain number of repetitions n is necessary. If, on the average, the repetitions are equally spaced and of the same duration and intensity, then n is proportional to the time t of learning. Hence

$$t = u(h), \tag{4}$$

and, inversely,

$$h = \bar{u}(t), \tag{5}$$

when $\bar{u}$ is the inverse function of u. The total number of circuits that are conditioned at a time t is equal to

$$N(t) = \int_0^{\bar{u}(t)} N(h)\,dh; \tag{6}$$

or, introducing expression (5) into (6) and denoting by $N_1(t)$ the result of substitution of expression (5) into (3), we have

$$N(t) = \int_0^t N_1(t)\,dt\,. \tag{7}$$

This expression is a measure of R_c.

We may understand the effects of shocks if we make the plausible assumption that the circuits with highest h, which were the last to become permanently excited, are the first to be affected by the shocks, who bring them back into the unexcited state. The effect of the shocks is to "unlearn" all responses. If after a time t of learning, m shocks have been administered, expression (6) is reduced by an amount which is a function of both m and $N_1(t)$, and increases with m. Graphically, this amounts to "moving back" along the curves OA and $O'B$, and thus cutting down the ordinate. If we start, let us say, at the point t_3, and consider arbitrarily the beginning of the psychosis as corresponding to the point C of intersection of OA and $O'A'$, we see that the longer the duration of the disease, the farther back we must go. Hence the longer the duration, the more shocks should be given. If all individuals were equally susceptible to shocks and alike in other respects, the necessary number m of shocks to effect a cure would be simply an increasing function of the duration T of the psychosis

$$m = v(T). \tag{8}$$

Actually, the necessary value of m will fluctuate around the value given by equation (8) according to some distribution function, so that for a given T the probability of m being between m and $m + dm$ is given by

$$M(T, m)\,dm; \quad \int_0^\infty M(T, m)\,dm = 1\,, \tag{9}$$

where for a fixed T the value of m which maximizes $M(T, m)$ is given by equation (8). If m shocks are given to every individual of

a population, characterized by a given T, the percentage of cures will be

$$P = \int_0^m M(T, m)\, dm\,. \tag{10}$$

Because of equation (9), the P, m curve raises monotonically to an asymptotic value which decreases with T.

According to this picture, the function $M(T, m)$ can be directly determined graphically from the experimental curves discussed by L. Danziger and H. D. Landahl (1945). As to the relationship

$$P_{\text{asymp.}} = r(T), \tag{11}$$

this would be determined largely by the function (3) and by the mechanism which we assume about the effect of the shocks in bringing the circuits back into the unexcited state.

The general characteristics of the experimental curves are correctly represented by the theory outlined here. An exact comparison will require our making explicit assumptions about the functions $N(h)$ and $v(T)$. The function $u(h)$ is determined from theoretical considerations (Rashevsky, 1938). We may try to determine those functions in such a way as to make a good agreement between theory and experiment. But at present the experimental data are not sufficiently accurate for that purpose.

Since $R_c(t) \propto N(t)$, therefore it follows from equation (7) that *formally* we may write

$$\frac{dR_c}{dt} = N_{1c}(t)\,, \tag{12}$$

where the index $1c$ refers to conditioned pathways for the correct response.

If we consider the learning curves as representing frequencies c of correct responses, then we have a different type of equation, of the form (Landahl, 1941b)

$$\frac{dc}{dt} = S(c, t)\,. \tag{13}$$

Now let us consider a more complex situation, namely, that the two responses represented by the curves OA and $O'B$ not only cross-inhibit each other, but that each response inhibits permanently the conditioning pathways of the other. Thus as R_w increases in strength, R_c is being "unlearned". Instead of equation (12) we now have

$$\frac{dR_c}{dt} = N_{1c}(t) - F_1(R_w)\ ;$$
$$\frac{dR_w}{dt} = N_{1w}(t) - F_2(R_c)\,. \tag{14}$$

Instead of equation (13) we shall have, denoting by w the incidence of the wrong response,

$$\frac{dc}{dt} = S(c,t) - U(w)\,;$$
$$\frac{dw}{dt} = S_1(w,t) - U_1(w)\,. \tag{15}$$

The time curve of the incidences of normal and abnormal behavior is now determined by either equations (14) or (15), provided we know the explicit form of the functions involved.

The important thing in this generalization is that now it is possible for the curve OA to begin to *decrease* after the onset of the disease, while the curve $O'B$ keeps increasing. The application of shocks will bring down the intensity of R_w. But if the disease has lasted long enough, the intensity R_c may be so small that even the "normal" behavior will be "unlearned", and the individual will not respond always normally unless adequately re-educated.

The considerations given above suggest also that besides the duration of the psychosis, there may be another parameter which determines the Danziger-Landahl curves, namely, the *age* at the onset of the disease. The earlier the onset, the less completely the "normal" behavior was learned, and the more it will be deteriorated after a given duration of the illness.

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LITERATURE

Danziger, Lewis and H. D. Landahl. 1945. "Some Quantitative Aspects of Shock Therapy in Psychoses." *Bull. Math. Biophysics*, 7, 213–218.

Householder, A. S. and H. D. Landahl. 1945. *Mathematical Biophysics of the Central Nervous System.* Bloomington: The Principia Press.

Landahl, H. D. 1938. "A Contribution to the Mathematical Biophysics of Psychophysical Discrimination." *Psychometrika*, 3, 107–125.

Landahl, H. D. 1941a. "Studies in the Mathematical Biophysics of Discrimination and Conditioning: I." *Bull, Math Biophysics*, 3, 13–26.

Landahl, H. D. 1941b. "Studies in the Mathematical Biophysics of Discrimination and Conditioning: II." *Bull. Math. Biophysics*, 3, 63–69.

Landahl, H. D. 1943. "Studies in the Mathematical Biophysics of Discrimination and Conditioning: III." *Bull. Math. Biophysics*, 5, 103–110.

Lettvin, J. Y. and W. Pitts. 1943. "A Mathematical Theory of the Affective Psychoses." *Bull. Math. Biophysics*, 5, 139–148.

Rashevsky, N. 1938. *Mathematical Biophysics*. Chicago: The University of Chicago Press.

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A PROBLEM IN THE MATHEMATICAL BIOPHYSICS OF INTERACTION OF TWO OR MORE INDIVIDUALS WHICH MAY BE OF INTEREST IN MATHEMATICAL SOCIOLOGY

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The behavior of an individual may be discussed in terms of a "satisfaction function". An individual may be considered to always behave in such a way as to make his satisfaction function a maximum. The interaction of two individuals may consist in a cooperation in the production of any kind of objects of satisfaction. Those objects may be either material goods or anything else. The satisfaction of each individual is determined by his share in the total output as well as by the effort he makes. It is shown that for a prescribed method of sharing a behavior in which each individual attempts to maximize the total satisfaction of both individuals results in a greater output than a behavior in which each individual attempts to maximize his own satisfaction.

When an individual possesses some goods which he likes, or performs a pleasant activity, or perceives something pleasant, it is commonly accepted that the more of the pleasant goods he has, or the more of the pleasant activity he performs, the more satisfied he is. The fundamental question is whether the notion of satisfaction can be given a quantitative definition, and whether satisfaction can be measured directly or indirectly. An affirmative answer to both those questions has been given by L. L. Thurstone (1931). Thurstone arrives at the conclusion that in many cases the satisfaction function varies as the logarithm of the "object of satisfaction" which the individuals possess.

The notion of satisfaction function becomes especially useful if we introduce the rather plausible assumption that every individual adjusts his behavior in such a way as to maximize his satisfaction. Combining this assumption with a proper mathematical expression for the satisfaction function, we should be able to make quantitative predictions about the behavior of the individual.

Let an individual perform an amount of work y, producing an amount of objects of satisfaction $x = ay$. These objects may represent material goods, or such less tangible things as the enjoyment of music or scientific discovery. Let the satisfaction function S with respect to x be of the form $A \log \alpha x$ (Thurstone, 1931). We shall

assume that S decreases linearly with increasing y. This amounts to assuming that work is always unpleasant. Actually, the variation of S with y is more likely to be such that it first increases to a positive maximum, then decreases, becoming negative. Such an assumption would complicate our calculations. The more restrictive assumption made here is not likely to introduce serious limitations. We thus have

$$S = A \log \alpha x - By \, , \tag{1}$$

where A and B are constants. Because if $x = ay$, we have

$$S = A \log \alpha ay - By \, . \tag{2}$$

This has a maximum for

$$y = A/B \, . \tag{3}$$

If the individual tries to make his satisfaction function a maximum, he will perform the amount $y = A/B$ of work, and produce the amount $x = aA/B$ of objects of satisfaction.

Consider two individuals, with satisfaction functions of the same form, performing respectively the amounts y_1 and y_2 of work, and producing the amounts

$$x_1 = a_1 y_1 ; \quad x_2 = a_2 y_2 \tag{4}$$

of objects of satisfaction, when working separately. Their satisfaction functions are

$$S_1 = A_1 \log \alpha_1 x_1 - B_1 y_1 ; \tag{5}$$

$$S_2 = A_2 \log \alpha_2 x_2 - B_2 y_2 \, . \tag{6}$$

When the individuals work on the production of the same objects together, their productivity in general increases. The total amount $x_1 + x_2$ of objects produced will be greater than the sum of $a_1 y_1 + a_2 y_2$. Depending on the nature of the work and on the type of cooperation, we shall have different expressions for $x_1 + x_2$. We shall here consider one of the simplest possible assumptions, namely

$$x_1 + x_2 = a_1 y_1 + a_2 y_2 + a_3 y_1 y_2 \, . \tag{7}$$

For $y_2 = 0$, this reduces to $a_1 y_1$; for $y_1 = 0$, it reduces to $a_2 y_2$.

The total amount $x_1 + x_2$ of objects produced may be distributed in different ways between the two individuals. If they apply the principle that a person receives in proportion to the *amount* of work he does, then the total amount $x_1 + x_2$ will be divided between the two individuals in the ratio y_1/y_2 . If a person receives in proportion to the *effectiveness* of his work, then the total amount $x_1 + x_2$ will be

divided in the ratio a_1y_1/a_2y_2. Finally, the division may be made in a fixed ratio, independent of the relative amounts of work. This ratio may in particular be equal to 1.

We shall first consider the first case, that of remuneration in proportion to effectiveness of work. The other case is handled in a similar way, and leads to similar results.
We have:

$$\frac{x_1}{x_1 + x_2} = \frac{a_1y_1}{a_1y_1 + a_2y_2}. \tag{8}$$

Equations (7) and (8) give:

$$x_1 = \frac{a_1y_1 + a_2y_2 + a_3y_1y_2}{a_1y_1 + a_2y_2}\, a_1y_1; \tag{9}$$

$$x_2 = \frac{a_1y_1 + a_2y_2 + a_3y_1y_3}{a_1y_1 + a_2y_2}\, a_2y_2. \tag{10}$$

Introducing those expressions into equations (5) and (6), we obtain

$$S_1 = A_1 \log \alpha_1 \frac{a_1y_1 + a_2y_2 + a_3y_1y_2}{a_1y_1 + a_2y_2}\, a_1y_1 - B_1y_1; \tag{11}$$

$$S_2 = A_2 \log \alpha_2 \frac{a_1y_1 + a_2y_2 + a_3y_1y_2}{a_1y_1 + a_2y_2}\, a_2y_2 - B_2y_2. \tag{12}$$

The amount of work which each individual will perform will now be determined by his attitude towards the satisfaction function. Two possibilities exist:

1) Each individual tries to maximize his own satisfaction function, regardless of the other individual. In this case the values of y_1 and y_2 are determined from

$$\frac{\partial S_1}{\partial y_1} = 0; \quad \frac{\partial S_2}{\partial y_2} = 0. \tag{13}$$

2) Each individual tries to maximize the total satisfaction $S = S_1 + S_2$. In this case y_1 and y_2 are determined from

$$\frac{\partial S}{\partial y_1} = 0; \quad \frac{\partial S}{\partial y_2} = 0. \tag{14}$$

From equation (11) we have:

$$\frac{\partial S_1}{\partial y_1} = \tag{15}$$

$$A_1 \frac{a_1^3 y_1^2 + a_1^2 a_3 y_1^2 y_2 + 2a_1^2 a_2 y_1 y_2 + a_1 a_2^2 y_2^2 + 2a_1 a_2 a_3 y_1 y_2^2}{(a_1 y_1 + a_2 y_2)(a_1 y_1 + a_2 y_2 + a_3 y_1 y_2) a_1 y_1} - B_1 ,$$

and from equation (12), we find a similar expression for $\partial S_2/\partial y_2$.

Introducing expression (15), and a corresponding one for $\partial S_2/\partial y_2$ into equations (13), gives us two cubic equations in y_1 and y_2, whose solution is unpractical. We can, however, investigate the properties of the solution, without actually solving the equations.

Put

$$\frac{\partial S_1}{\partial y_1} = F_1(y_1, y_2) , \tag{16}$$

and consider the curve determined by

$$F_1(y_1, y_2) = 0 . \tag{17}$$

If $y_2 = 0$, then, from equation (15), we have:

$$F_1(y_1, 0) = \frac{A_1}{y_1} - B_1 . \tag{18}$$

Hence, for $y_2 = 0$, the requirement $F_1(y_1, y_2) = 0$ leads to

$$y_1 = \frac{A_1}{B_1} . \tag{19}$$

Thus the curve $F_1(y_1, y_2) = 0$ intersects the y_1-axis at the point given by equation (19) (Figure 1).

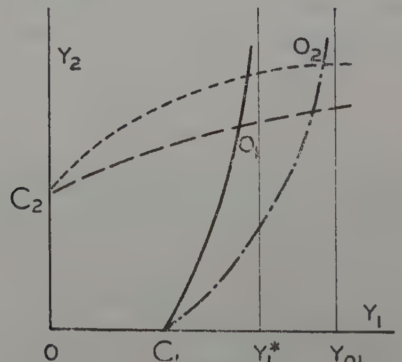

FIGURE 1

When $y_2 = \infty$, then, from equations (15) and (16), we have

$$F(y_1, \infty) = A_1 \frac{a_1a_2^2 - 2a_1a_2a_3y_1}{(a_1a_2^2 - a_1a_2a_3y_1)\,y_1} - B_1. \tag{20}$$

Putting $F_1(y_1, \infty) = 0$ and introducing the notation

$$\frac{A_1}{B_1} = C_1, \tag{21}$$

gives

$$a_1a_2a_3y_1^2 + (a_1a_2^2 - 2C_1a_1a_2a_3)\,y_1 - C_1a_1a_2^2 = 0. \tag{22}$$

Equation (22) has two roots:

$$\begin{aligned} y_1^* &= \frac{-a_1a_2^2 + 2C_1a_1a_2a_3 + \sqrt{a_1^2a_2^4 + 4(C_1a_1a_2a_3)^2}}{2a_1a_2a_3}; \\ y_1^{**} &= \frac{-a_1a_2^2 + 2C_1a_1a_2a_3 - \sqrt{a_1^2a_2^4 + 4(C_1a_1a_2a_3)^2}}{2a_1a_2a_3}. \end{aligned} \tag{23}$$

Since $y_1^* > 0$, $y_1^{**} < 0$, the second root has no physical meaning, and we consider therefore only y_1^*. Hence, the curve $F_1(y_1, y_2) = 0$ is such that $y_1 = \infty$ for $y_1 = y_1^*$. We have, from the first equation (23),

$$y_1^* - C_1 = \frac{-a_1a_2^2 + \sqrt{(a_1a_2^2)^2 + 4(C_1a_1a_2a_3)^2}}{2a_1a_2a_3} > 0. \tag{24}$$

Hence

$$C_1 < y_1^*. \tag{25}$$

We now shall prove that the curve $F_1(y_1, y_2) = 0$ is monotonic. To this end we must establish that $dy_1/dy_2 > 0$. We have

$$\frac{dy_1}{dy_2} = -\frac{\dfrac{\partial F_1}{\partial y_2}}{\dfrac{\partial F_1}{\partial y_1}}. \tag{26}$$

The function F_1 is given by the right side of equation (15). The derivative $\partial F_1/\partial y_1$ is of the form D_1/D, where D is the square of the denominator in equation (15), and is therefore always positive. The derivative $\partial F_1/\partial y_2$ is of the form D_2/D. Evaluating D_1 and D_2, we find after laborious but elementary calculation, that D_1 consists of a sum of only negative terms, while D_2 consists of a sum of positive

terms. Hence $D_1 < 0$, $D_2 > 0$. Therefore $\partial F_1/\partial y_1 < 0$; $\partial F_1/\partial y_2 > 0$. Hence, from equation (26) it follows that $dy_1/dy_2 > 0$.

By similar reasoning we determine the general properties of the curve $\partial S_2/\partial y_2 = F_2(y_1, y_2) = 0$. (Figure 1, broken line). The two curves intersect at only one point, O_1. The coordinates y_1 and y_2 of O_1 represent the solution of the system (13).

Now let us investigate case 2), which leads to equations (14). From equations (9) and (10) we have:

$$\frac{\partial x_1}{\partial y_2} = \frac{a_1{}^2 a_3 y_1{}^2}{(a_1 y_1 + a_2 y_2)^2} > 0; \quad \frac{\partial x_2}{\partial y_1} = \frac{a_2{}^2 a_3 y_2{}^2}{(a_1 y_1 + a_2 y_2)^2} > 0. \tag{27}$$

Hence

$$\frac{\partial S_1}{\partial y_2} = \frac{A_1}{\alpha_1 x_1} \frac{\partial x_1}{\partial y_2} > 0; \quad \frac{\partial S_2}{\partial y_1} = \frac{A_2}{\alpha_2 x_2} \frac{\partial x_2}{\partial y_1} > 0. \tag{28}$$

Equations (14) may be written:

$$\frac{\partial S_1}{\partial y_1} + \frac{\partial S_2}{\partial y_1} = 0; \quad \frac{\partial S_1}{\partial y_2} + \frac{\partial S_2}{\partial y_2} = 0. \tag{29}$$

Since (equation 28) $\partial S_2/\partial y_1 > 0$, therefore $\partial S/\partial y_1 > 0$ at all points of the full curve of Figure 1. Since for sufficiently large values of y_1, $\partial S_2/\partial y_1$ tends to zero, as seen from equations (24) and (28), while $\partial S_1/\partial y_1$ tends to a negative constant, therefore the locus of the maximum of S will be to the right of the full line (alternate line). By a similar argument, we prove that the locus of $\partial S/\partial y_2 = 0$ is given by the dotted line of Figure 1. The values of y_1 and y_2, which correspond to a maximum of $S = S_1 + S_2$, are now given by the point O_2. Hence in this case, y_1 and y_2, and therefore x_1 and x_2 are greater than in the previous case.

The case of a division in the output in a fixed ratio is treated in a similar way, and the conclusions are the same.

If we attempt to generalize those considerations to N individuals, we may use the same method, considering now instead of lines in the y_1, y_2 plane, hypersurfaces in a hyperspace. Instead of expression (7), we now shall have

$$\Sigma x_i = \sum_i a_i y_i + \sum_i \sum_k a_{ik} y_i y_k. \tag{30}$$

The proof that maximizing S leads to a greater output than maximizing the S_i's separately depends, in the two-dimensional case, on the inequalities $\partial S_1/\partial y_2 > 0$, $\partial S_2/\partial y_1 > 0$ (expressions 28). In the N-dimensional case we must have

$$\partial S_i/\partial y_k > 0 \, . \tag{31}$$

It can be shown, however, that with expression (30), inequality (31) is not necessarily satisfied. Therefore our conclusion cannot be generalized. In order to be able to generalize it, we must assume that cooperation increases the output more strongly than given by expression (30).

An expression of the form

$$\sum x_i = \sum_i a_i y_i + \sum_i \sum_k a_{ik} y_i^2 y_k + \sum_i \sum_k b_{ik} y_i y_k^2 \tag{32}$$

will lead to inequality (31). For expressions of that type our conclusions hold for any number N of individuals.

If in expression (7) we assume $a_3 < 0$, or if we assume some of the a_{ik} and b_{ik} in expression (32) to be negative, we have the case of *conflict* of two or several individuals, instead of cooperation. The conflict *reduces* the total output. It is interesting to study in a similar way as above the various aspects of behavior in conflict.

A more detailed presentation of this study will appear in a forthcoming book by the author (Rashevsky, 1947).

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LITERATURE

Rashevsky, N. 1947. (In press.) *Mathematical Theory of Human Relations. Sociology, and History as Rational Mathematical Sciences.* Bloomington, Indiana: The Principia Press.

Thurstone, L. L. 1931. "The Indifference Function." *Jour. Soc. Psychol.*, 2, 139–167.

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MATHEMATICAL THEORY OF MOTIVATION INTERACTIONS OF TWO INDIVIDUALS: I

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The behavior of two individuals is considered as consisting of an increase or a decrease of productive output. Motivation for increase is the derivative of a "satisfaction function". This is an algebraic sum of the well-known Thurstone satisfaction curve and another essentially negative quantity, which is a product of a "reluctance parameter" and the "effort". Each individual attempts to maximize his own total satisfaction. The resulting behavior is examined under a variety of conditions; namely, 1) equal sharing of produce without prescribed sharing of effort; 2) various contracts prescribing the sharing of effort; 3) situations in which one individual is more aware of the underlying motivations than the other. It is these latter situations which under the simplest assumptions of equal sharing, without prescribed sharing of effort, lead to parasitism, i.e. total cessation of effort on the part of one individual. This happens when one individual becomes aware of the other's *automatic* adjustment of his effort so as to bring about a total optimum output, which is a constant. Parasitism is prevented by various forms of contracts in which either the effort necessary for the total optimum output is shared according to a prescribed ratio or the effort of one individual is fixed *a priori* as a function of the effort of the other. In the latter case the respective efforts become a function of a single variable, and each of the satisfaction functions is maximized by a particular value or values of this variable. In general, these critical values do not coincide for the two satisfaction functions. The problem of finding forms of contract which will result in identical critical (maximizing) values of the variable for both satisfaction functions leads to a functional equation.

Introduction

Consider the motivations of an individual so drastically simplified that they can be reduced to two quantities, R and E, which combined describe the individual's *satisfaction.*

We take R to be essentially positive and representing the contribution to satisfaction by the reward received, and E essentially negative and representing the detraction from satisfaction because of effort expended. Following N. Rashevsky (1947) and L. L. Thurstone (1931) we suppose R to be a logarithmic function of remuneration and E a linear function of the effort.

We take R to be log $(1 + x)$ where x is the measure of the re-

ward received. This differs from the Thurstone satisfaction function, namely,

$$s = k \log x + c$$

by choice of units and of the origin of the coordinate system. Where Thurstone arbitrarily fixes the origin of *satisfaction,* i.e., the X-axis, we arbitrarily fix the origin of the *reward,* namely the Y-axis, so that $R(0) = 0$. This is equivalent to considering a certain minimum level of "subsistence" resulting in zero satisfaction. We do not consider situations with negative satisfactions (pain). The reason for our choice of origin is that we wish to discuss conditions which are necessary for an individual to start working *at all.* Situations are conceivable where an individual is assured a certain minimum of subsistence without the necessity of working, if by working we understand an activity which results in at least a temporary accumulation, to provide for some stretch of the future, if only for a day. Thus the Hebrews in the desert gathering their daily manna from day to day were not working according to our definition, but when they gathered a double portion on Friday to provide for the Sabbath (when no manna fell), they were working.

We have furthermore reduced as many as possible of the proportionality constants to unity, since nothing is assumed in this paper about the units by which the various quantities are measured, e.g. effort, output, reward, satisfaction, reluctance, etc. Hence quantities which are considered proportional to each other are taken to be equal, e.g. effort and output. The mathematical treatment is simplified thereby. If occasion should arise where an experimental procedure is indicated, it would be easy to insert the necessary parameters in the formulae.

Thus "output" represents simply elements contributed to a certain reservoir out of which the "rewards", i.e. elements which give rise to satisfactions are distributed. Output may consist of bananas, steel ingots, paintings, or scientific papers, while the rewards may correspondingly represent available energy, increased technological potential, intensity of esthetic experience, or academic promotions.

Our problem is to investigate the behavior of two individuals as affected by various forms of the satisfaction function. The constants of the equations may be considered as parameters. These parameters are peculiar to the individuals and to the relationship that exists between them (e.g. how the remuneration and the effort is distributed among them). The problem may be varied by introducing relationships between the variables and the parameters themselves. Finally an interesting situation results if the concept of *initiative* is intro-

duced, that is if one of the individuals acts *first* on certain assumptions, while the second individual then must consider the actions of the first as part of *his* motivation.

The fundamental variables will be x and y, representing the respective efforts expended in production by two individuals X and Y. Their output is proportional to their efforts, and the "units" are so chosen that the output is set equal to the effort. Thus x and y represent both rates of work and rates of output. All other quantities will be treated as parameters, in general constant for a particular situation.

We consider now the functions $S_1(x, y)$ and $S_2(x, y)$, representing individual satisfactions derived by X and Y respectively as a result of their efforts:

$$
\begin{aligned}
&S_i(x, y) = R_i(x, y) + E_i(x, y), \\
&R_i \geqq 0, \; E_i \leqq 0, \\
&x \geqq 0, y \geqq 0 \; (i = 1, 2).
\end{aligned}
\tag{1}
$$

Our inquiry considers conditions under which one or both individuals will be motivated to produce *together* instead of separately, motivation being always towards increased individual satisfaction. We shall also consider certain relations between X and Y which may or may not (depending on relations between certain parameters) lead to parasitism, i.e. a situation in which one individual utilizes part of the produce of the other without himself producing.

As we shall see, the positions of the maxima of the individual satisfaction functions will depend not only on the variables and the parameters and the form of the function, which is obvious from the mathematical set up, but *also on certain judgments* made by the individual themselves. For example, nothing is said *a priori* about the relationship that may exist between x and y. But the behavior of each individual does depend on the assumptions which he chooses to make about the existence of such relationships. In some cases such relationships will appear to him to flow out of his own and his neighbor's motivations. He can, for example, regard x and y as *independent* variables. In this case, his attempts to maximize his individual satisfaction will be represented by setting its *partial* derivative with respect to the appropriate variable equal to zero. Or he can assume any number of relations which *fix* y as a function of x. In this case his behavior will be represented by a total derivative of the satisfaction function set equal to zero. Finally it may happen that he begins by considering x and y as independent variables, and a relation between x and y then flows out of one maximized satisfaction function, which relationship he can then re-substitute into his satis-

faction function and maximize by total differentiation.

It may happen that the two individuals make different assumptions and/or exhibit different behavior. This behavior may then be interpreted as desires to bring about one or another situation. If the situations are incompatible, we are led to conflicts of the type studied by N. Rashevsky (1947). Another problem that flows out is then the study of conditions under which equilibria (agreements) can be reached. The stability of such equilibria can also be studied.

I

Case I. *The individual produces and consumes alone*

In our simplest case, the effort of an individual will depend on a single parameter β, the constant of proportionality connecting the function E with effort. We are dropping the subscripts since we are dealing with only one individual. We have

$$E = -\beta x . \tag{2}$$

Thus

$$S = \log(1 + x) - \beta x . \tag{3}$$

We have chosen our satisfaction function in such a way that $S(0) = 0$. We shall refer to β as the *reluctance.*

Lemma 1. *If the satisfaction of an individual producing and consuming alone is given by equation* (3), *he must have* $\beta < 1$ *in order to produce at all.*

Proof. Since $S(0) = 0$, for production to start, we must have $S'(0) > 0$,

$$S'(0) = 1 - \beta . \tag{4}$$

Hence $\beta < 1$.

Suppose, therefore, $\beta < 1$. Then if the individual attempts to maximize his satisfaction function,

$$\frac{dS}{dx} = \frac{1}{1+x} - \beta = 0 ,$$

$$x = \frac{1}{\beta} - 1 , \tag{5}$$

$$S_{\max}(\beta) = \beta - 1 - \log \beta ,$$

which is positive for $\beta < 1$. Now $S'_{\max} = 1 - 1/\beta < 0$ for $0 < \beta < 1$, and $S_{\max}$ increases with decreasing β. Hence

Lemma 2. *Individuals with smaller reluctances will achieve larger maximal satisfactions.*

We shall constantly compare this maximized satisfaction of the individual producing alone with the satisfactions arising from various situations where two individuals produce together and share their output. We shall thus be seeking conditions under which it *pays* for the individual to associate himself with another, (i.e. his satisfaction increases thereby).

Case II. *The individuals* X *and* Y *have equal reluctances and share their produce equally.*

In this case we have

$$S_1 = \log\left(1 + \frac{x+y}{2}\right) - \beta x,$$
$$S_2 = \log\left(1 + \frac{x+y}{2}\right) - \beta y. \tag{6}$$

If now each individual acts so as to attempt to maximize his satisfaction function *without regard for any dependence of* y *upon* x, we can deduce the conditions on β for production to start. We have

$$\frac{\partial S_1}{\partial x} = \frac{1}{2+x+y} - \beta; \quad \left.\frac{\partial S_1}{\partial x}\right|_{\substack{x=0\\y=0}} = \frac{1}{2} - \beta. \tag{7}$$

This is positive if and only if $\beta < 1/2$.

Lemma 3. *To produce at all under assumptions* (6), *it is necessary* that $\beta < 1/2$.

Let us now see what the relations between x and y become *as viewed by* X, who tries to maximize his individual satisfaction function and considers y as being constant. We have

$$\frac{\partial S_1}{\partial x} = \frac{1}{2+x+y} - \beta = 0,$$
$$x = 1/\beta - 2 - y. \tag{8}$$

The individual X will consider $(x + y) = 1/\beta - 2$ as the optimum of the total production; in other words, he will adjust his output to that of y according to equations (8). Under these conditions no motivation exists to increase the total output beyond the prescribed constant optimum. If Y produces all of it, X will produce nothing, and vice versa. We have, of course, no opportunity to observe a society even vaguely resembling the schematic representation used here; but we do ob-

serve situations where people will work just sufficiently to keep "body and soul" together. Such conditions are sometimes observed in colonial countries, where the only motivations are hunger on one hand and a reluctance to work on the other.

Substituting the value of x in equations (8) into (6), we see that X's maximum satisfaction is a monotone decreasing function of x:

$$S_1 = \log \frac{1}{2\beta} - \beta x. \tag{9}$$

Since we consider only non-negative values of the variables, S_1 will be greatest under assumptions (8) when $x = 0$.

It must be again emphasized that equation (9) follows from the assumption that X maximizes his satisfaction while considering y as a constant. Only in this way it seems to X that under conditions (6) it does not pay for him to work. This conclusion really rests on both the kind of assumptions X makes and also on the fact that it is X who presumably controls the situation. X argues in effect in this manner: "Y will want to maximize his satisfaction. Under conditions (6), his satisfaction will be a maximum for $x + y = 1/\beta - 2$. Whatever I do, he will bring the amount of produce to that total. If I do nothing, he will do it all. My own satisfaction under these conditions is greatest if I do nothing." Of course Y can argue this way equally well. But if they both do nothing, they starve if the output happens to be food. Eventually one or the other or both will yield to some extent. Thus a *contract* may be established which fixes the relationship between the amount of work done by X and Y *in advance.*

Case III. *The individual* X *agrees to produce the* k*th part of the optimum output.*

We note that under conditions (6) and by maximizing each satisfaction function under the assumption that the other variable is constant, X and Y can agree on what the *total* optimum output should be; that is to say, since each gets one-half of his total, he will not work any more if he is guaranteed that remuneration. Having agreed on the total output, let us suppose that they now fix the ratio of dividing the necessary labor. We are interested in knowing under what conditions X will decide to join this arrangement rather than produce alone as in Case I. Denoting by S^*_1 the maximum value of S_1 in this special case, we have

$$x=\frac{k}{\beta}(1-2\beta)\,,\quad y=\frac{1-k}{\beta}(1-2\beta)\,,$$
$$S^*_1=\log\frac{1}{2\beta}-k(1-2\beta)\,. \tag{10}$$

Comparing this value of S_1 with the value of S_{max} of Case I, we have the difference

$$S^*_1-S_{max}=1-\beta-\log 2-k(1-2\beta)\,. \tag{11}$$

This difference will be greater than zero if

$$k<\frac{1-\beta-\log 2}{1-2\beta}\,. \tag{12}$$

The expression on the right is $< 1/2$ for all values of β. Hence X will not join under equal distribution of labor. We also see that for very small β, $k \sim 1-\log 2$. As β approaches $1-\log 2$, k approaches zero monotonically, since the derivative with respect to β of the right side of relation (12) is less than zero. For very large β, the upper limit of k is $1/2$. However, this last situation has no meaning in our case because for production to take place at all, β must be less than $1/2$. We summarize the results on motivation for production and sharing in

Theorem 1. *Under the forms of the satisfaction function expressed by equations* (3) *and* (6), *the following conditions on the reluctance govern the motivations*:

For $\beta \geqq 1$, *no production will take place.*

For $1/2 \leqq \beta < 1$, *production will take place on individual basis only.*

For $1-\log 2 \leqq \beta < 1/2$, *production is possible under equal sharing, but will not pay for either individual.*

For $\beta < 1-\log 2$, *a sufficient motivation for* X *to join* Y *under equations* (6) *is* (12).

Case IV. *The individual* X *will receive the* mth *part of the optimum output, and the labor necessary for optimum production will be shared equally.*

We now have:

$$S^*_1 = \log\left(1 + \frac{m(1-2\beta)}{2\beta}\right) - \frac{1-2\beta}{2}, \quad m \leqq 1,$$

$$S^*_1 - S_{\max} = \log\left(1 + \frac{m(1-2\beta)}{2\beta}\right) - \frac{1-2\beta}{2} - \beta + 1 + \log\beta \qquad (13)$$

$$= \log\left(\frac{2\beta + m(1-2\beta)}{2}\right) + \frac{1}{2}.$$

Since $1 - 2\beta > 0$, this expression grows monotonically with m. The upper limit on m is 1. Evaluating expressions (13) for $m = 1$, we have $S^*_1 - S_{\max} = \log(1/2) + 1/2$, which is never positive.

Hence even if X were to take the entire output (labor being shared equally), his satisfaction under the joint arrangement is decreased compared with his satisfaction under individual production. We thus have established

Theorem 2. *When individual satisfactions are maximized under the assumption that the other variable is constant, a larger share in the optimum output under equal sharing of labor is not sufficient inducement for joint production.*

The reason for the disadvantage under conditions (6) is clear when one compares equations (5) and (8). For a given β the *total* "optimum" production resulting from conditions (6) is actually less than the individual production resulting from equation (3).

Case V. *The individuals agree in advance on the relationship* $y = kx$.

Here for the first time we are dealing with a situation where X has *advance* knowledge as to how Y's output will depend on his own. Until now, he was using deductions based on Y's presumed maximizing of his own satisfaction function. Now X maximizes his own satisfaction by total differentiation:

$$S_1 = \log\left(1 + \frac{(1+k)x}{2}\right) - \beta x,$$

$$dS_1/dx = \frac{1+k}{2+x+kx} - \beta = 0, \qquad (14)$$

$$1 + k - 2\beta - \beta x - \beta k x = 0.$$

Hence

$$x = \frac{1 + k - 2\beta}{\beta + \beta k},$$

and

$$y = \frac{k + k^2 - 2\beta k}{\beta + \beta k}. \tag{15}$$

Therefore we have

$$\frac{x+y}{2} = \frac{1 + 2k + k^2 - 2\beta - 2\beta k}{2\beta + 2\beta k};$$

and

$$1 + \frac{x+y}{2} = \frac{1+k}{2\beta}. \tag{16}$$

We now find

$$\begin{aligned} S^*_1 &= \log\left(\frac{1+k}{2\beta}\right) - \frac{1+k-2\beta}{1+k} \\ &= \log(1+k) - \log 2 - \log\beta - 1 + \frac{2\beta}{1+k}, \end{aligned} \tag{17}$$

$$\begin{aligned} S^*_1 - S_{\max} &= \log(1+k) - \log 2 - \beta + \frac{2\beta}{1+k} \\ &= \log(1+k) - \log 2 + \frac{\beta - \beta k}{1+k}. \end{aligned}$$

Let us show that this difference is greater than zero if and only if $k > 1$ (assuming $k > 0$).

If $F(k) = \log\left(\frac{1+k}{2}\right) + \frac{\beta(1-k)}{1+k}$, then

$F(1) = 0$,

$$F'(k) = \frac{1}{1+k} - \frac{2\beta}{(1+k)^2},$$

$F'(1) > 0; F'(k) > 0$ for $0 < k < 1$.

Hence $F(k)$ is monotone increasing with k in this interval and has a root at $k = 1$.

Case V therefore gives the following result:

Theorem 3. *Under a ratio of production agreed upon in advance, it will pay* X *to join if* Y *agrees to produce proportionately to* X *with the constant of proportionality* k > 1.

Similarly we can prove

Theorem 4. *Under an agreement to produce equally, it will pay* X *to join if he is to receive a fraction of the optimum output which exceeds* 1/2 .

Comparing the conclusions of these theorems with those of Theorems 1 and 2, we see that considerably greater inducement in division of labor is required for joining if no guarantee is made about the dependability of Y's output, while no inducement involving a greater share in the output is sufficient to bring about a partnership. The difference between partial and total differentiation may be interpreted psychologically as a difference in confidence in the partner's output as a function of one's own. Unless that confidence manifests itself (either in the form of an explicit contract or in other forms, as we shall see below), there will be no incentive for combined production and sharing of the output.

II

Forms of Contract with Common Maxima of the Satisfaction Function

So far we have viewed the question of cooperation and sharing from the point of view of one of the individuals. Presumably, he does all the calculating and decides whether or not it pays for him to join the other under certain conditions. It is clear from Theorems 3 and 4 that in situations where it pays for X to join, it does not pay Y. This follows by the symmetry of the situation. Furthermore, X fixes the output of both individuals according to equations (15) and (16). These values do not necessarily correspond to what Y considers the optimum. The two do not coincide unless $k = 1$.

The question arises: do there exist forms of contract other than the trivial one, $y = x$, where the maxima of satisfactions occur for the same values of x for both individuals? We shall not give here the most general form of such a contract, but will show that an infinity of such non-trivial *contracts of equal optima* exists.

We seek a function $y = y(x)$ such that any $\bar{x}$ satisfying

$$S'_1(x) = \frac{1 + y'(x)}{2 + x + y(x)} - \beta = 0$$

will also satisfy

$$S'_2(x) = \frac{1 + y'(x)}{2 + x + y(x)} - \beta\, y'(x) = 0.$$

Obviously, a necessary and sufficient condition on y is that $y'(\bar{x}) = 1$. A trivial solution of this functional equation is $y = x + c$, where c = constant. But it is by no means necessary that $y'(x)$ be *identically* equal to unity as in that solution. In fact, consider $y = Ax^2 + C$. We wish $y'(\bar{x}) = 2A\bar{x} = 1$. Thus $\bar{x} = 1/2A$. Then

$$\frac{2}{2 + \frac{1}{2A} + \frac{1}{4A} + C} = \beta; \quad A = \frac{3\beta}{8 - 8\beta - 4\beta C}.$$

Thus A or C can be determined.

We shall not find the most general form of the contract which gives common maxima, but simply state:

Theorem 5. *There exist an infinite number of contracts other than* $y = x + c$, *such that any* $\bar{x}$ *which maximizes* S_2 *also maximizes* S_1 *and conversely.*

If we weaken the restriction of "and conversely" in Theorem 5 we are led to an even greater manifold of contracts. One such special case we shall consider here, namely where every x is a "maximizing" $\bar{x}$ for S_1, so that Theorem 6 holds trivially without the "and conversely".

Case VI. *Contract:* S_1 *is independent of* x. X's *output is then determined so as to maximize* S_2.

Let $y = y(x)$ be the form of the contract such that $S_1(x)$ is constant.

Then if

$$S'_1(x) = \frac{1 + y'(x)}{2 + x + y(x)} - \beta = 0 \tag{18}$$

identically in x, we have

$$y = Ce^{\beta x} - x - 2, \tag{19}$$

where C is a constant of integration. Substituting this function into $S_1(x, y)$, we get

$$S^*_1 = \log(C/2) = \text{constant}. \tag{20}$$

Let us now maximize $S_2(x) = \log\ (C/2) - \beta(Ce^{\beta x} - x - 2) + \beta x$. We have

$$\frac{\partial S_2}{\partial x} = -\beta^2 C e^{\beta x} + 2\beta = 0, \tag{21}$$

$$x = \frac{1}{\beta} \log \frac{2}{\beta C}.$$

Hence

$$\begin{aligned} S^*_2 &= \log \frac{1}{\beta} - 2 + \log \frac{2}{\beta C} + 2\beta \\ &= \log \frac{2}{\beta^2 C} - 2 + 2\beta > 0 \end{aligned} \tag{22}$$

for sufficiently small values of β and/or C.

We have here a very special case of the general problem considered above, since every x is a "maximizing" value for S_1. We see that it will pay X to join such an arrangement for C sufficiently large. But if C is large, it may not pay Y.

For such an agreement to be profitable to both, we must have

$$\log (C/2) > \beta - 1 - \log \beta, \tag{23}$$

and

$$\log 2 - 2 \log \beta - \log C - 2 + 2\beta > \beta - 1 - \log \beta. \tag{24}$$

Adding inequalities (23) and (24), we obtain

$$2\beta - 2 - 2 \log \beta > 2\beta - 2 - 2 \log \beta,$$

a contradiction.

Therefore the contract determined by equation (19) is not sufficient inducement for sharing the joint output.

We can, however, prove the rather obvious

Theroem 6. *Let* C $= \frac{2}{\beta} e^{\beta - 1}$. *Then the satisfactions of* X *and* Y *will be identical with those of Case I.*

A generalization of Case VI would be finding a form of contract which would make S_1 some arbitrarily prescribed function of x, and investigating the resulting changes in the respective satisfaction functions.

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LITERATURE

Rashevsky, N. 1947. "A Problem in Mathematical Biophysics of Interaction of Two or More Individuals, which May Be of Interest in Mathematical Sociology." *Bull. Math. Biophysics*, 9, 9–15.

Thurstone, L. L. 1931. "The Indifference Function." *Jour. Social Psychol*, 2, 139–167.

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THE MECHANISM OF THE MIDDLE EAR: I. THE TWO PISTON PROBLEM

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The drum, ossicles and oval window of the middle ear can be considered equivalent to a system of two pistons connected by a lever, where the leverage results not only from the combined motion of the three ossicles, but also from the non-rigid fibrated structure of the drum. This paper studies the transmission of sound from air to liquid by means of such a system and finds conditions on the dimensions, mass and elasticity of the system which should be met in a "good" ear. Two methods, based on different conceptions of "good," are used, and they give similar results. Most of these results are in good quantitative agreement with actual data.

Introduction. The function of the middle ear is to transform the sound vibrations of the air in the outer ear into vibrations of the liquid in the inner ear. As will be seen, vibrations in a liquid have generally much smaller amplitudes, but much larger variations in pressure, than vibrations in air. In its transmission of vibrations from air to liquid, the middle ear must therefore produce a considerable reduction in amplitudes and a considerable increase in forces.

The middle ear consists of three main parts: (1) the drum, a movable surface in contact with the air of the outer ear, which receives the air vibrations, (2) the oval window, a movable surface in contact with the liquid (endolymph) of the inner ear, which imparts vibrations to this liquid, and (3) the ossicles, a system of levers transmitting the motions of the drum into motions of the oval window. The ossicles are connected to the drum by the handle of the hammer and to the oval window by the footplate of the stirrup.

We shall, as a first approximation, consider the drum and the oval window to be rigid pistons moving in long cylinders. In fact, however, the drum is a thin membrane, 1/10 mm thick (Bethe, 1928, page 411), inclined at 45 degrees to the auditory canal; and the oval window is rigid, but has a rotary movement rather than a movement of translation. In a subsequent paper we shall establish the equivalence of the drum to a piston whose displacement is greater than the displacement of the handle of the hammer and on which acts an elastic force proportional to its deviation from the position of equilibrium.

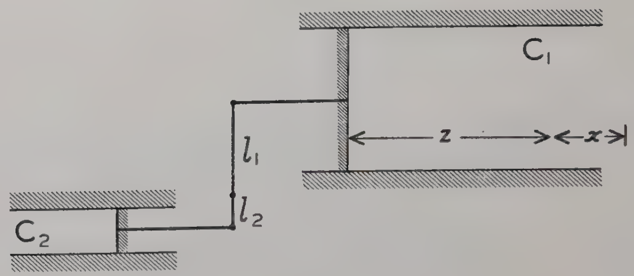

FIGURE 1

The two-piston problem. We consider two cylinders C_1 and C_2, (Figure 1) representing the outer and the inner ear respectively. The two cylinders have different cross-sections and contain different fluids. Each cylinder is closed by a piston and the two pistons are connected by a lever. The pressure on the back side of each piston will be considered constant and equal to the average pressure on the front side. A sound wave travels down the cylinder C_1, imparts a motion to the pistons, and thus creates a wave in the cylinder C_2. Our purpose is to compare the two waves, and to determine the dimensions of the pistons and lever arms so as to produce in C_2 a wave of maximum energy for a given energy of the original wave in C_1.

Notations. To all symbols, except t, defined below, the subscript one or two will be added according to which of the cylinders C_1 or C_2 is considered:

$t =$ time
$z =$ distance of a considered particle of fluid from the piston, when the fluid and the piston are at rest
$x(t, z) =$ distance of a particle of fluid from its rest position
$X(t) = x(t, 0) =$ position of piston
$\lambda =$ coefficient of elasticity, defined by $p = p_0 - \lambda(u - u_0)/u_0$, where u indicates a volume of fluid
$\delta =$ specific gravity of fluid
$v =$ velocity of propagation of sound
$s =$ surface of piston
$m =$ mass of piston
$l =$ length of lever arm
$\phi =$ force exerted by the lever on the piston
$-kX(t) =$ elastic force acting on the piston.

Mechanics of one fluid. We consider the section of fluid between z and $z + \Delta z$. Its volume at rest is $s \Delta z$ and at an arbitrary moment is $s(\Delta z + \Delta x)$. Therefore, by definition of λ, we have

$$p = p_0 - \lambda \frac{s(\Delta z + \Delta x) - s \Delta z}{s \Delta z} = p_0 - \lambda \frac{\Delta x}{\Delta z}.$$

Passing to the limit, we obtain

$$p(t, z) = p_0 - \lambda \frac{\partial x(t, z)}{\partial z}. \tag{1}$$

Let $\partial^2 x/\partial t^2$ be the average acceleration of the different particles of fluid in the section between z and $z + \Delta z$. The equation of motion of this section is

$$\delta s \Delta z \frac{\partial^2 x}{\partial t^2} = -s p(t, z + \Delta z) + s p(t, z) = -s \Delta p(t, z).$$

Therefore, $\delta \dfrac{\partial^2 x}{\partial t^2} = - \dfrac{\partial p}{\partial z}.$

Using also equation (1), we obtain

$$\delta \frac{\partial^2 x(t, z)}{\partial t^2} = \lambda \frac{\partial^2 x(t, z)}{\partial z^2}. \tag{2}$$

The general solution of this partial differential equation is

$$x(t, z) = f\left(t + \sqrt{\frac{\delta}{\lambda}} z\right) + g\left(t - \sqrt{\frac{\delta}{\lambda}} z\right), \tag{3}$$

where f and g are arbitrary differentiable functions.

If we make $g = 0$ and consider any constant z_0, then $x(t, z)$ keep the constant value $f(\sqrt{\delta/\lambda}\, z_0)$ at the point $z = z_0 - \sqrt{\lambda/\delta}\, t$. Therefore f represents a wave moving towards the piston with the velocity

$$v = \sqrt{\frac{\lambda}{\delta}}. \tag{4}$$

Similarly, g represents a wave moving away from the piston with the same velocity $v = \sqrt{\lambda/\delta}$.

Reflection on the piston. Setting $z = 0$ in equation (3), we have

$$X(t) = x(t, 0) = f(t) + g(t). \tag{5}$$

This equation determines the reflected wave $g(t)$ in terms of the incident wave $f(t)$ and of the motion of the piston $X(t)$. Substitution into equation (3) thus gives, with the help of equation (4),

$$x(t,z)=f\left(t+\frac{z}{v}\right)-f\left(t-\frac{z}{v}\right)+X\left(t-\frac{z}{v}\right). \tag{6}$$

Using equations (6), (1), and (4), we calculate the pressure on the piston:

$$p(t,0)=p_0+v\delta\,[X'(t)-2f'(t)]\,. \tag{7}$$

We suppose at present that the pressure on the back side of the piston is constant and equal to p_0. Variations of the pressure on the back side will be considered later. The equation of motion of the piston is

$$\begin{aligned} m\,X''(t) &= \phi-k\,X(t)-(p-p_0)s \\ &= \phi-k\,X(t)-sv\delta[X'(t)-2f'(t)]. \end{aligned} \tag{8}$$

System of two pistons. The lever connection between the two pistons gives

$$\begin{aligned} &\frac{X_1(t)}{l_1}=\frac{X_2(t)}{l_2}=X(t)\,; \\ &l_1\,\phi_1(t)+l_2\,\phi_2(t)=0\,. \end{aligned} \tag{9}$$

The quantity $X(t)$ is introduced in order to maintain a symmetry of notations in our calculations. Elimination of ϕ_1, ϕ_2, X_1, X_2 from equations (8) for each piston and equations (9) gives

$$\begin{aligned} (m_1l_1^2+m_2l_2^2)\,X''(t) &= -(k_1l_1^2+k_2l_2^2)X(t) \\ &\quad -(s_1l_1^2v_1\delta_1+s_2l_2^2v_2\delta_2)\,X'(t) \\ &\quad +2[s_1l_1v_1\delta_1f'_1(t)+s_2l_2v_2\delta_2f'_2(t)]\,. \end{aligned}$$

We introduce the notations:

$$\begin{aligned} M &= m_1l_1^2+m_2l_2^2 \\ K &= k_1l_1^2+k_2l_2^2 \\ N &= s_1l_1^2v_1\delta_1+s_2l_2^2v_2\delta_2 \\ h(t) &= 2[s_1l_1v_1\delta_1f_1(t)+s_2l_2v_2\delta_2f_2(t)], \end{aligned} \tag{10}$$

and obtain thus

$$M\,X''(t)+N\,X'(t)+K\,X(t)=h'(t). \tag{11}$$

We add the boundary conditions:

$$X(-\infty)\neq\infty,\; X'(-\infty)\neq\infty\,. \tag{12}$$

Equation (11) with conditions (12) determine $X(t)$ and thus the motion of the pistons. The reflected waves $g_1(t)$ and $g_2(t)$ will then be determined by equations (5) and (9). We have, indeed, when we suppose that no wave moves toward the piston in the cylinder C_2:

$$\begin{aligned} f_2(t) &= 0, & h(t) &= 2s_1 l_1 v_1 \delta_1 f_1(t), \\ g_2(t) &= l_2 X(t), & g_1(t) &= l_1 X(t) - f_1(t). \end{aligned} \tag{13}$$

Thus in particular, the function $g_2(t)$, in which we are mainly interested, is proportional to the solution $X(t)$ of equations (11) and (12).

Sinusoidal wave. We shall first discuss the case where $h(t)$ is a sine function: $h(t) = A \sin \omega t$. Then it can easily be verified that the solution of equations (11) and (12) is

$$X(t) = A \frac{\cos \omega t_0}{N} \sin \omega (t - t_0), \tag{14}$$

where

$$\tan \omega t_0 = \frac{M\omega - K/\omega}{N}. \tag{15}$$

For a given amplitude A of h, the amplitude of X reaches the maximum A/N for $\omega^2 = K/M$.

If the equation of the oncoming wave is

$$f_1(t) = B \sin \omega t \tag{16}$$

then, by equations (13) and (14), the outgoing wave $g_2(t)$ is

$$g_2(t) = \frac{2\, s_1 l_1 l_2 v_1 \delta_1}{N} B \cos \omega t_0 \sin \omega (t - t_0). \tag{17}$$

Let us compare the energies carried by the waves f_1 and g_2. The energy F_1 carried by the wave $f_1 = B \sin \omega t$ through any point z per unit time is given (Stevens and Davis, 1938, page 28) by the formula

$$F_1 = \tfrac{1}{2} s_1 B^2 \omega^2 v_1 \delta_1. \tag{18}$$

The energy G_2 carried by the wave g_2 of equation (17) is

$$\begin{aligned} G_2 &= \tfrac{1}{2} s_2 \left[2 \frac{s_1 l_1 l_2 v_1 \delta_1}{N} B \cos \omega t_0 \right]^2 \omega^2 v_2 \delta_2 \\ &= F_1 s_2 s_1 \left[2 \frac{l_1 l_2 v_1 \delta_1}{N} \cos \omega t_0 \right]^2 \frac{v_2 \delta_2}{v_1 \delta_1}. \end{aligned}$$

We introduce the notations:

$$P = \frac{v_2\delta_2}{v_1\delta_1}, \quad Q = \frac{s_1 l_1^2}{s_2 l_2^2}. \tag{19}$$

Using these notations and the value of N given in equation (10), we obtain

$$G_2 = F_1 \frac{4\,P/Q}{(1+P/Q)^2} \cos^2 \omega t_0 .$$

Using also equation (15) we find

$$\frac{G_2}{F_1} = \frac{4}{2 + P/Q + Q/P} \, \frac{1}{1 + [(M/N)\omega - (K/N)/\omega]^2}. \tag{20}$$

Efficiency. The preceding equation is the main equation of this paper. The quantities related are:

G_2/F_1, which represents the efficiency of the system
P, which depends on the two fluids
Q, which represents the geometrical proportions of the system
M/N, which depends on the mass of the system
K/N, which depends on the elasticity of the system
ω, which describes the wave transmitted.

In order to maximize the efficienty G_2/F_1, we must minimize $P/Q + Q/P$ and $|(M/N)\omega - (K/N)/\omega|$. The quantity $P/Q + Q/P$ is large when P and Q have different orders of magnitude and has a a minimum for

$$P = Q. \tag{21}$$

In order to minimize $|(M/N)\omega - (K/N)/\omega|$ for a large range of ω, we must make both M/N and K/N as small as possible.

For $P = Q$ and $\omega^2 = K/M$ we obtain an efficiency equal to one.

We thus have found how to maximize the efficiency given by equation (20). It is interesting to notice for what range of values the efficiency does not differ appreciably from its maximum value one. This happens when the variable terms in the denominators of equation (20) do not exceed the constant terms in these denominators. Denoting by $n_1 = \omega_1/2\pi$ and $n_2 = \omega_2/2\pi$ the lowest and highest frequencies which are easily audible, we obtain the approximate conditions

$$\begin{gathered} \tfrac{1}{2} \leqq \frac{P}{Q} \leqq 2, \\ M/N \leqq 1/(2\pi n_2), \quad K/N \leqq 2\pi n_1 . \end{gathered} \tag{22}$$

Numerical comparisons. When the fluid in C_1 is air and in C_2 is water, we have (Hodgman, 1945)

$$v_1 = 340\,, \quad v_2 = 1500 \text{ meters per second,}$$
$$\delta_1 = .0013\,, \; \delta_2 = 1 \quad \text{grams per cubic centimeter.}$$

Therefore $P = 3400$.

The areas of the drum and of the oval window are (Stevens and Davis, 1938, page 260):

$$s_1 = 90 \text{ mm}^2; \quad s_2 = 3.2 \text{ mm}^2.$$

The ossicles of the ear form a system of levers the reduction ratio of which has been estimated (Beatty, 1932, page 6) to be 3. Measurements (Helmholtz, 1873, page 57) indicate that the average displacement of the drum is three times as large as the displacement of the handle of the hammer. We have thus approximately

$$\frac{s_1}{s_2} = \frac{90}{3.2} = 28; \quad \frac{l_1}{l_2} = 3 \times 3 = 9; \quad Q = 28 \times 9^2 = 2270.$$

Therefore $P/Q = 3400/2270 = 3/2$. This value satisfies the double inequality (22).

We shall see now that the inequality (22) which involves the mass M is satisfied. The area of the drum being 90 mm² and its thickness 1/10 mm, its mass will be 10 mg. Observation of pictures (Helmholtz, 1875; figure 36) shows that the volume of the ossicles and oval window certainly does not exceed 50 mm³. Their mass therefore is not more than 50 mg. Considering the average displacement of the ossicles to be 5 times smaller than the displacement of the drum, we find from equation (10):

$$M \leqq (10 + 50/5^2)\, l_1^2 = 12\, l_1^2.$$

Using equations (10), (19) and (21), we find:

$$N = 2\, s_1\, l_1^2\, v_1\, \delta_1 = 2 \times 90 \times 340\,000 \times .0013\, l_1^2 = 80\,000\, l_1^2.$$

Therefore $M/N = 1/6600$ and the second inequality (22) is satisfied for frequencies of 1000 or less.

We cannot study the last inequality (22) for it is difficult to estimate the factor K. Some remarks on this inequality will be made later however.

If the drum and ossicles are removed, only an energy equal to $(s_2/s_1)F_1$ will reach the outer surface of the oval window. This energy will be transmitted to the endolymph with an efficiency $G_2/(s_2/s_1)F_1$ which we calculate by setting $Q = 1$, $M = K = 0$ in equation (20). We obtain:

$$\frac{G_2}{F_1}=\frac{s_2}{s_1}\frac{4}{2+P+1/P}=\frac{3.2}{90}\frac{4}{3402}=\frac{1}{24{,}000}.$$

The removal of drum and ossicles has reduced the efficiency from unity to the value 1/2400. The threshold of hearing is therefore elevated by an energy ratio of 24,000, or by an amount of 10 log 24000 = 44 decibels. Empirical data show that the hearing loss in such cases is 30 to 65 decibels (Stevens and Davis, 1938, page 253).

Non-periodical waves. The theory given above is obtained by assuming the sound wave to be periodical and of infinite duration. In practice, however, sounds are either non-periodic (noise) or periodic (musical note), but of a short duration. We shall show that results similar to inequalities (22) can be derived from the study of non-periodical waves.

We shall first show that the solution of equations (11) and (12) is

$$X(t)=\frac{-1}{M\beta}\int_{-\infty}^{0}e^{\alpha s}\sin\beta s\,d\,h(t+s)\,, \tag{23}$$

where

$$2\,M\,\alpha=N\,,2M\,\beta=(4\,K\,M-N^2)^{1/2}. \tag{24}$$

We put $u(t)=(M\,\beta)^{-1}\,e^{\alpha t}\sin\beta\,t$.
This function $u(t)$ satisfies the relations

$$\begin{gathered}u(-\infty)=u'(-\infty)=u(0)=0\,,u'(0)=M^{-1}\\ M\,u''(t)-N\,u'(t)+k\,u(t)=0\,.\end{gathered} \tag{25}$$

Substituting into equation (23), integrating by parts and differentiating, we obtain successively:

$$X(t)=\int_{-\infty}^{0}u(s)\,d\,h(t+s)=-u(s)h(t+s)\Big|_{-\infty}^{0}+\int_{-\infty}^{0}h(t+s)\,d\,u(s)$$

$$=\int_{-\infty}^{0}u'(s)\,h(t+s)\,ds\,,$$

$$X'(t)=\int_{-\infty}^{0}u'(s)\,d\,h(t+s)=u'(s)\,h(t+s)\Big|_{-\infty}^{0}-\int_{-\infty}^{0}h(t+s)\,d\,u'(s)$$

$$=M^{-1}\,h(t)-\int_{-\infty}^{0}u''(s)\,h(t+s)\,ds\,,$$

$$X''(t)=M^{-1}\,h'(t)-\int_{-\infty}^{0}u''(s)\,d\,h(t+s).$$

We combine the preceding equations:

$$MX''(t)+NX'(t)+KX(t)=h'(t)$$
$$-\int_{-\infty}^{0}[Mu''(s)-Nu'(s)+Ku(s)]\,dh(t+s).$$

Using relation (25), we have shown that expression (23) is a solution of (11). It satisfies also the conditions (12).

Transforming equation (23) by means of equations (13), we obtain

$$g_2(t)=-2\frac{s_1l_1l_2v_1\delta_1}{M\beta}\int_{-\infty}^{0}e^{\alpha s}\sin\beta s\,df_1(t+s). \tag{26}$$

We shall obtain conditions analogous to inequalities (22) by taking

$$f_1(t)=\begin{cases}0 \text{ for } t<0\\ A \text{ for } t\geqq 0\end{cases}$$

and finding the conditions on l_1/l_2, M/N and K/N for which the response $g_2(t)$ is rapidly damped and most intense.

Equation (26) gives, for $t\geqq 0$,

$$g_2(t)=2\frac{s_1l_1l_2v_1\delta_1}{M}\frac{A}{\beta}e^{-\alpha t}\sin\beta t. \tag{27}$$

In order to have a rapid damping, $2\alpha=N/M$ must be large. We thus obtain a condition similar to the second inequality (22).

We have approximately,

$$g_2(t)=\begin{cases}2\dfrac{s_1l_1l_2v_1\delta_1}{M}A\,e^{-\alpha t}\,t & \text{for } t\leqq\dfrac{\pi}{4\beta}\\[2ex] 2\dfrac{s_1l_1l_2v_1\delta_1}{M}\dfrac{A}{\beta}e^{-\alpha t} & \text{for } \dfrac{\pi}{4\beta}\leqq t\leqq\dfrac{3\pi}{4\beta}.\end{cases}$$

The maximum of the exact value of $g_2(t)$, given by equation (27), is obtained when $\tan\beta t=\beta/\alpha$. If $\beta\leqq\alpha$, this maximum falls in the region where the approximate value of $g_2(t)$ is independent of β. If $\beta>\alpha$, this maximum falls in the region where $g_2(t)$ is a rapidly decreasing function of β. Therefore, in order to obtain large values for $g_2(t)$, we must take $\beta<\alpha$, and little is gained by further decreasing β. The maximum B of $g_2(t)$ is then obtained for approximately $t=1/\alpha$ and equals approximately

$$B=2\frac{s_1l_1l_2v_1\delta_1}{M}\frac{A}{e\alpha}. \tag{28}$$

Using equations (24), we find that inequality $\beta \leqq \alpha$ gives $4KM - N^2 \leqq N^2$, therefore $K/N \leqq N/(2M) = \alpha$. We thus obtain a condition similar to the last inequality (22).

We shall maximize B as a function of l_1/l_2. From equations (28), (24), and (10), we have

$$B = \frac{4A}{e} \frac{s_1 l_1 l_2 v_1 \delta_1}{s_1 l_1^2 v_1 \delta_1 + s_2 l_2^2 v_2 \delta_2}.$$

This expression has a maximum $B = \frac{2}{e} \frac{l_2}{l_2} A$ for

$$\left(\frac{l_1}{l_2} \right)^2 = \frac{s_2 v_2 \delta_2}{s_1 v_1 \delta_1}.$$

We thus obtain again equation (21).

Remarks. We made our calculations supposing that the discriminant $N^2 - 4KM$ is negative and thus introducing trigonometric functions, because it is generally agreed that the ear is not critically damped. It has a natural frequency $\beta/2\pi$ contained between 550 and 1500 vibrations per second and a damping factor equal to about one-half of the critical damping, $\alpha = \frac{1}{2} \sqrt{K/M}$ (Stevens and Davis, 1938, p. 262).

Using inequalities (22), we find:

$$N^2 - 4KM \geqq N^2 (1 - 4\, n_1/n_2).$$

The discriminant $N^2 - 4KM$ being negative, we must have $n_2 < 4n_1$. Thus the range $n_1\, n_2$ is less than two octaves, and cannot be referred to as the range of easily audible frequencies, as was the case with inequalities (22).

Effect of the air in the middle ear. We have, up to now, supposed that the pressure at the back side of our piston is constant. This assumption is justified when, as for the oval window, the fluid back of the piston has a much smaller specific gravity δ and sound velocity v than the fluid in front. If however, as for the drum, the same fluid is found on both sides, we must modify our calculations as is done, for instance, in the following two cases.

I) This case will be studied briefly because it is of little interest for practical applications. It is the case where at the back of the piston, the fluid extends throughout an infinite cylinder, and where no wave moves in it towards the piston. Then, similar to equation (7), the pressure at the back of the piston is $p = p_0 - v\,\delta\,X'$. Equation (8) becomes

$$mX''(t) = \phi - kX(t) - s\,v\,\delta\,[2X'(t) - 2f'(t)].$$

The values for N and G_2/F_1, originally given by equations (10) and (20), are now

$$N = 2\, s_1 l_1^2 v_1 \delta_1 + s_2 l_2^2 v_2 \delta_2 ,$$

$$\frac{G_2}{F_1} = \frac{2}{2 + P/(2Q) + 2Q/P} \, \frac{1}{1 + [(M/N)\omega - (K/N)/\omega]^2} .$$

We have a maximum efficiency for $P = 2Q$. This maximum efficiency is at most equal to $1/2$.

II) We suppose in this case that the cylinder is closed by an irregularly shaped rigid partition at a small average distance L behind the piston. By small we mean that $v/(\omega L)$ equals at least several units ($2\pi v/\omega$ is the wave length of the sound transmitted). In this case, the waves are reflected from the piston, from the partition and from the sides of the cylinders so often that some uniformity of pressure will be maintained in the fluid.

By definition of λ and by equation (4), the average pressure is

$$p = p_0 - \lambda\, X/L = p_0 - v^2\, \delta\, X/L .$$

The average acceleration of the fluid is $X''/2$. Therefore the difference of pressure on the piston and on the partition is $-L\, \delta\, X''/2$. The pressure on the piston is therefore

$$p(t) = p_0 - v\, \delta\, X/L - L\, \delta\, X''/4 .$$

Equation (8) becomes

$$\begin{aligned} mX''(t) &= \phi - kX(t) \\ &\quad - s\, v\, \delta \left[\frac{v}{L} X(t) + X'(t) + \frac{L}{4v} X''(t) - 2f'(t) \right] . \end{aligned} \tag{29}$$

When $X(t) = A \sin \omega t$, the three first terms of the brackets in equation (29) have the magnitudes $\dfrac{v}{\omega L} A\omega$, $A\omega$ and $\dfrac{\omega L}{v}\dfrac{A\omega}{4}$. We can therefore disregard the last term, and also remark that the corrective term $(v/L)\, X(t)$ is larger than the corrective term $X'(t)$ found in case I.

Equation (11) becomes

$$MX''(t) + NX'(t) + (K + l_1^2 s_1 v_1^2 \delta/L)\, X(t) = h'(t). \tag{30}$$

As we have, from equations (10) and (21), $N = 2\, s_1 l_1^2 v_1 \delta_1$, equation (30) becomes

$$MX''(t) + NX'(t) + [K + v_1 N/(2L)]\, X(t) = h'(t).$$

Thus the effect of the air in the middle ear is to make a contribution $v_1/(2L)$ to the factor K/N.

Numerical calculations. The volume of the middle ear is 1 to 2 cc (Stevens and Davis, 1938, p. 249) and the area of the drum is 0.9 cm^2. Therefore $L = 1.5/0.9 = 1.4$ cm., and $L\ \omega/v = 2\ \pi\ Ln/34000 = n/4000$. The highest note played by the piano is about $n = 4000$. We are therefore justified in assuming $L\omega/v$ to be small, except for very high notes.

The contribution $v_1/2L$ to the factor K/N is $34000/2.8 = 12000 = 2\ \pi(2000)$. This contribution is extremely large, considering that the last inequality (22) must be satisfied, and requires that K be negative.

LITERATURE

Beatty, R. T. 1932. *Hearing In Man and Animals.* London: G. Bell and Sons.

Bethe, A. 1928. *Handbuch der Normalen und Pathologischen Physiologie.* Volume 11. Berlin: J. Springer.

Helmholtz, H. 1873. *The Mechanism of the Ossicles and Membrana Tempani.* New York: W. Wood & Co.

Helmholtz, H. 1875. *The Sensation of Tone as a Physiological Basis of Music.* London: Longmans Green & Co.

Hodgman, M. S. 1945. *Handbook of Chemistry and Physics.* Cleveland: Chemical Rubber Publishing Co.

Stevens, S. S. and H. Davis. 1938. *Hearing, Its Psychology and Physiology.* New York: John Wiley and Sons.